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APPROXIMATING LENGTHS, AREAS AND VOLUMES BY POLYGONS AND POLYHE--ETC(U)

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ABSTRACT

Let C be a circle and P a convex polygon inscribed in C such that the center O of C is inside P. Let P^* be the convex polygon obtained from P by adding on each side s of P a triangle of base s and vertex V_s such that $V_s \in C$, and OV_s is perpendicular to s. To judge how well Length P and Area P approximate the Length C and Area C, respectively, we use the two functionals

$$\lambda(P) = \text{Length } P/\text{Length } C \quad \text{and} \quad \alpha(P) = \text{Area } P/\text{Area } C .$$

<u>Theorem 1</u>. $\alpha(P) < \lambda(P)$ <u>and</u> $\lambda(P) = \alpha(\lambda^*)$.

In words: P approximates Length C better than it approximates Area C. To achieve the same quality of approximation for areas, as for lengths, we have to pass from P to P^*.

A similar situation is shown to hold in space. Let S be a sphere and Π a convex polyhedron inscribed in S, such that Π contains the center O of S. Let Π^* be the polyhedron derived from Π by erecting on each face f of Π a pyramid of base f and vertex V_f, such that $V_f \in S$ and such that the segment OV_f is perpendicular to the face f. Now we use the functionals

$$a(\Pi) = \text{Area } \Pi/\text{Area } S \quad \text{and} \quad v(\Pi) = \text{Vol } \Pi/\text{Vol } S .$$

<u>Theorem 2.</u> $v(\Pi) < a(\Pi)$ <u>and</u> $a(\Pi) = v(\Pi^*)$.

Inspecting a table giving the numerical values of $a(\Pi)$ and $v(\Pi)$, where Π runs through the five regular solids, we conclude from Theorem 2, that O^* and I^* are not convex, where O and I denote the regular tetrahedron and icosahedron, respectively. This is again confirmed by Theorem 3 which gives the necessary and sufficient conditions for Π^* to be convex, for the case when Π is a regular pyramid.

AMS (MOS) Subject Classifications: 50-01, 50A30

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APPROXIMATING LENGTHS, AREAS AND VOLUMES BY

POLYGONS AND POLYHEDRA

I. J. Schoenberg

A conversation with E. V. Schenkman and Edward Silverman concerning the biblical value of π led to the problem discussed here.

Introduction

Let C be a circle of radius $R = 1$ and let P_n denote a regular polygon of n sides inscribed in C. The relations

$$\text{Length } C = 2\pi, \quad \text{Length } P_6 = 6 , \tag{1}$$

show that the perimeter of the regular hexagon already gives the biblical approximation 3 to π (I. Kings VII, 23).

If we pass to areas we find that

$$\text{Area } C = \pi, \quad \text{Area } P_{12} = 3 . \tag{2}$$

Thus in terms of areas, we have to go to the regular dodecagon P_{12} to obtain an equally good approximation to π.

The rather vague inference that we draw from this remark is that areas are not as easily approximated as lengths by means of inscribed polygons. However, before pursuing this hunch any further, we present the best thing that we have to offer in this note, and this is Kürschak's proof of the second relation (2) (see [2]). Kürschak's paper suggested to us the present note.

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Figure 1 shows P_{12} whose center is O, while BCDE is the circumscribed square of side 2. Let F be the circumcenter of the triangle OA_1A_6 and Figure 1 also shows all the other eleven similar circumcenters. Next we triangulate the big square as shown. Kūrschak observes next that we have only triangles of two kinds: The equilateral triangles like A_1A_6F, $A_6BA_{11}, \ldots,$ and isosceles triangles like OA_1F, OA_6F, $A_1BA_6, \ldots$. Moreover, all equilateral triangles (their number is 16) are congruent among themselves, and so are the isosceles triangles (there are 32 such) are congruent among themselves. This is easily seen if we derive our entire diagrams by first constructing the regular starred dodecagon

$$A_1A_2A_3 \ldots A_{12} .$$

Further details may be omitted.

Now imagine all the triangles within P_{12} to be made of separate pieces of cardboard. Remove the 9 triangles in P_{12} that are within the fourth quadrant A_1OA_{10}; there are 3 equilateral triangles and 6 isosceles triangles. These 9 triangles are now used to fill in the three empty areas at B, C, D, that are outside P_{12} and within the big square. The three unit squares OA_1BA_4, OA_4CA_7, and OA_7DA_{10} are now completely covered by cardboard triangles that also covered P_{12}. We have therefore established the second relation (2). Figure 1 also makes an attractive design for a tile, especially if alternate triangles are shaded, or colored, as shown.

Kürschak's Tile

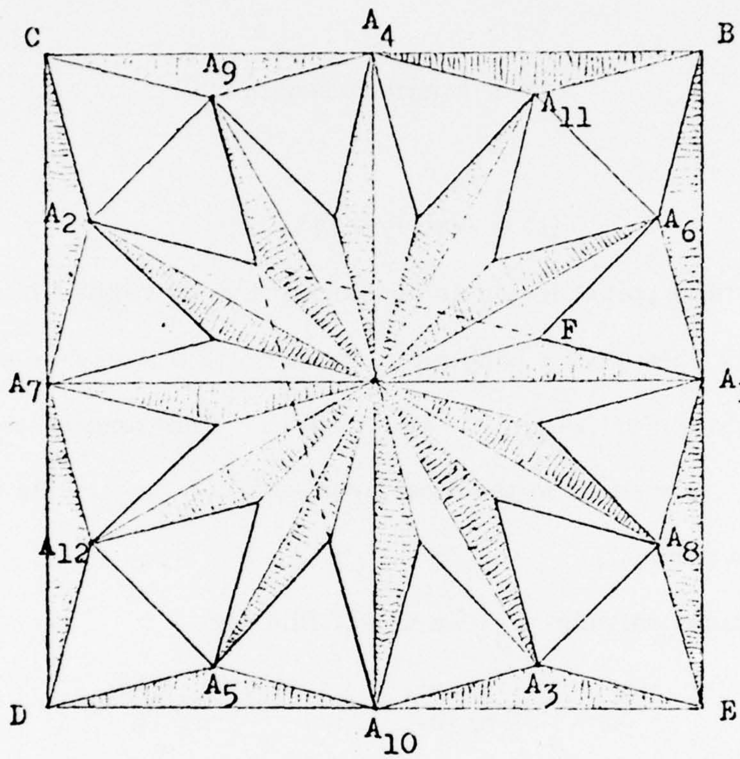

Fig.1

1. Lengths and areas in the plane

Let P denote a convex polygon inscribed in C, not necessarily regular, which is assumed to contain in its interior the center O of C. Our aim is to estimate how well the perimeter and area of P approximate to the perimeter and area of C. As measures of approximation we use the two ratios

$$\lambda(P) = \text{Length } P/\text{Length } C \tag{1.1}$$

and

$$\alpha(P) = \text{Area } P/\text{Area } C . \tag{1.2}$$

These are naturally proper fractions because of the convexity of P. Moreover $1 - \lambda(P) = (\text{Length } C - \text{Length } P)/\text{Length } C$ is the relative error of the approximation of Length C by Length P. This remark applies also to $\alpha(P)$, as well as to the other approximation functionals used throughout this note.

For a regular polygon P_n we easily find that

$$\lambda(P_n) = \sin\frac{\pi}{n}\Big/\left(\frac{\pi}{n}\right), \quad \alpha(P_n) = \sin\frac{2\pi}{n}\Big/\left(\frac{2\pi}{n}\right) \tag{1.3}$$

and in particular, that

$$\alpha(P_{2n}) = \lambda(P_n) . \tag{1.3'}$$

In words: <u>In order to obtain an approximation of the area of</u> C <u>by the area of a regular polygon</u> P_m, <u>which is as good as the approximation of the perimeter of</u> C <u>by the perimeter of</u> P_n, <u>we must go to a polygon with the double number</u> $m = 2n$ <u>of sides</u>. For $n = 6$ we find that

$$\alpha(P_{12}) = \lambda(P_6) = 3/\pi ,$$

which is Kürschak's remark.

The relation (1. 3) can be generalized as follows. We associate with the polygon P a new polygon P^* by means of the following construction: Let $P = p_1p_2 \ldots p_n$. On each side p_ip_{i+1} $(p_{n+1} = p_1)$ we drop the perpendicular OO_i, from the center O, and extend it beyond O_i until it meets C in the point q_i. We denote by P^* the polygon obtained by adding to P the n triangles $p_ip_{i+1}q_i$.

As an example we mention that $P_n^* = P_{2n}$. A generalization of the relation (1. 3') is the following

Theorem 1. The following relations hold

$$\alpha(P) < \lambda(P) \quad \text{and} \quad \lambda(P) = \alpha(P^*) . \tag{1.4}$$

Proof: We establish first the second relation (1. 4). Writing $a_i = p_ip_{i+1}$ and $r_i = OO_i$ we conclude from the definition of P^* that

$$\text{Area } P^* = \text{Area } P + \frac{1}{2} \sum_i (1 - r_i)a_i .$$

Since $\text{Area } P = \frac{1}{2} \Sigma\, r_ia_i$, we get that $\text{Area } P^* = \frac{1}{2}$ Length P or

$$\frac{\text{Area } P^*}{\pi} = \frac{\text{Length } P}{2\pi}$$

which is the second relation (1. 4).

Evidently $P^* \supset P$ implies that $\alpha(P^*) > \alpha(P)$, so that the second relation (1. 4) implies the first.

2. Areas and volumes in space

Let S be a sphere of radius $R = 1$ and let Π denote a convex polyhedron inscribed in S, i.e. having all of its vertices on the surface of S. A precise way of describing Π is as follows: Let $p_1, p_2, \ldots, p_n$ be distinct points on the surface of S. We may then define the polyhedron Π as the <u>convex hull</u> of the points p_i. We shall also assume that

(2.1) <u>the center</u> O <u>of</u> S <u>is in the interior of</u> Π .

We need one further restrictive assumption concerning the polyhedron Π. Let

$$F_i = q_1 q_2 \ldots q_s$$

be one of the faces of Π, and let π_i denote its plane. The polygon F_i is convex and inscribed in the circle C_i which is the intersection of π_i with the spherical surface S. Let O_i be the center of C_i; evidently the segment OO_i is perpendicular to the plane π_i. We shall assume that

(2.2) <u>the center</u> O_i <u>is in the interior of the face</u> F_i ,

and this for all faces of Π.

This assumption is evidently satisfied for each of the five regular polyhedra inscribed in S. If all faces of Π are <u>triangles</u>, then the above assumption is equivalent to the requirement that all the faces of Π should be <u>acute-angled</u> triangles.

For convenience we shall write

$$r_i = OO_i \, . \tag{2.3}$$

We consider the four quantities

$$\begin{aligned} &\text{Area } \Pi = \Sigma |F_i|, \quad \text{Area } S = 4\pi \\ &\text{Vol } \Pi = \frac{1}{3} \Sigma\, r_i |F_i|, \quad \text{Vol } S = \frac{1}{3} 4\pi \, , \end{aligned} \tag{2.4}$$

where $|F_i| = \text{Area } F_i$, and raise the question similar to the one discussed in §1 for the plane: <u>How does the approximation of</u> Area S <u>by</u> Area Π, <u>compare with the approximation of</u> Vol S <u>by</u> Vol Π ?

The analogues of the measures of approximation (1.1) and (1.2) are now the ratios

$$a(\Pi) = \frac{\text{Area } \Pi}{\text{Area } S} \tag{2.5}$$

and

$$v(\Pi) = \frac{\text{Vol } \Pi}{\text{Vol } S} \, . \tag{2.6}$$

We associate with the polyhedron Π a new polyhedron Π^* by means of the following construction: For each face F_i, of Π, we extend the segment OO_i beyond O_i until it meets the surface S in the point v_i. <u>We denote by</u> Π^* <u>the polyhedron obtained by adding to</u> Π <u>the pyramids having the vertex</u> v_i <u>and base</u> F_i, <u>and this for all faces of</u> Π. An example: If T is a <u>regular</u> tetrahedron inscribed in S, then T^* is easily seen to be a cube inscribed in S.

The space analogue of Theorem 1 is the following

Theorem 2. The following relations hold

$$v(\Pi) < a(\Pi) \quad \text{and} \quad a(\Pi) = v(\Pi^*) \ . \tag{2.7}$$

Proof: Again we establish first the second relation (2.7). From the definition of Π^* we obtain that

$$\text{Vol } \Pi^* = \text{Vol } \Pi + \frac{1}{3} \Sigma (1 - r_i) |F_i| \ .$$

Here Vol Π cancels the sum $\frac{1}{3} \Sigma r_i |F_i|$ and we obtain

$$\text{Vol } \Pi^* = \frac{1}{3} \text{Area } \Pi \ .$$

On dividing by $4\pi/3$ we get that

$$\frac{\text{Vol } \Pi^*}{4\pi/3} = \frac{\text{Area } \Pi}{4\pi}$$

which is the second relation (2.7).

Since $\Pi^* \supset \Pi$ evidently implies that $v(\Pi^*) > v(\Pi)$ we have that

$$a(\Pi) = v(\Pi^*) > v(\Pi)$$

and the inequality (2.7) is established.

3. How well do the five regular polyhedra approximate the sphere?

We wish to measure the approximation by means of the functionals (2.5) and (2.6). The algebraic values of the areas and volumes of the regular solids can be found e.g. in our reference [1, Table I, 292-293]. We list here the numerical values of $a(\Pi)$ and $v(\Pi)$ because we wish to compare their magnitudes. From this source the following table was compiled:

Table 1

Π	Symbol	# of vertices	# of faces	$a(\Pi)$	$v(\Pi)$
Tetrahedron	T	4	4	.36755	.12252
Cube	C	8	6	.63662	.36755
Octahedron	O	6	8	.55133	.31831
Dodecahedron	D	20	12	.83673	.66491
Icosahedron	I	12	20	.76192	.60546

The first information that we gather from Table 1 is that each of the five polyhedra approximates the area of S better than it approximates its volume. Indeed, notice that $a(\Pi) > v(\Pi)$ for all five polyhedra. This was expected in view of the first inequality (2.7) of Theorem 2.

Secondly, we notice that <u>the quality of the approximation arranges our polyhedra in the order</u> D, I, C, O, T, for if we use the symbol $f(\Pi)$ to denote either $a(\Pi)$ or $v(\Pi)$, we find that

$$f(D) > f(I) > f(C) > f(O) > f(T) .$$

Notice that for this ordering also the numbers of their vertices are in decreasing order: 20, 12, 8, 6, 4.

Further, perhaps unexpected, facts are the inequalities

$$a(T) > v(O) \quad \text{and} \quad a(C) > v(I) .$$

It does seem surprising indeed, that the 4-vertex <u>tetrahedron</u> should be more efficient in approximating Area S, as compared with the performance

of the 6-vertex octahedron in approximating Vol S. In the same sense is the 8-vertex cube more efficient than the 12-vertex icosahedron. In view of Theorem 2 the relation

$$a(T) = v(C) = .36755 (= 2\sqrt{3}/(3\pi))$$

does not surprise us, because as already mentioned $C = T^*$.

Next we observe that C and O are dual to each other. By this we mean that we can so place C and O, both inscribed in S, that the set of vertices of one, is identical with the set of vertices of the pyramids having as bases the faces of the other. It follows that C^* and O^* have the same set of $8 + 6 = 14$ vertices. However, this does not mean that the polyhedra C^* and O^* are identical. Rather we have the

Corollary 1. The polyhedron O^* is not convex.

Proof: O^* has the same 14 vertices as C^*, while C^* is convex. (This is intuitively evident and easily verified). If O^* were convex, it would have to be identical with C^* (a convex polyhedron is uniquely defined by its vertices!) and therefore we would have that

$$v(C^*) = v(O^*) \ . \tag{3.1}$$

On the other hand Theorem 2 and Table 1 show that in fact

$$v(C^*) = a(C) = .63662 \quad \text{and} \quad v(O^*) = a(O) = .55133 \ .$$

Thus $v(C^*) > v(O^*)$, contradicting the conclusion (3.1), and proving Corollary 1.

Also the polyhedra D and I are in the same dual relationship, and let us assume that they are so inscribed in S, that the vertices of

one are the vertices of the pyramids built on the faces of the other. Therefore D^* and I^* have the same set of $20 + 12 = 32$ vertices. As above we have

Corollary 2. The polyhedron I^* is not convex.

Proof: D^* is evidently convex. If also I^* were convex, we would have $D^* = I^*$, hence $v(D^*) = v(I^*)$, while Theorem 2 and Table 1 show that

$$v(D^*) = a(D) = .83673 > v(I^*) = a(I) = .76192 .$$

In the next and last section we investigate the convexity of Π^* for a special kind of convex polyhedra Π, the regular pyramids.

4. The case of the regular pyramid

New, and perhaps more direct, proofs of Corollaries 1 and 2 will follow from Theorem 3 below concerning the regular pyramid (see Remark 3 below). Let $\Pi_n = N P_0 P_1 \ldots P_{n-1}$ be a regular pyramid having n lateral faces. Its apex is the *North Pole* N *of the unit sphere* $S: x^2 + y^2 + z^2 = 1$ while its base $P_0 P_1 \ldots P_{n-1}$ is the regular n-gon inscribed in the parallel circle of colatitude α. This means that $\measuredangle NOP_0 = \alpha$. We disregard the base $P_0 \ldots P_{n-1}$, and consider its lateral surface Π'_n formed of n isosceles triangles $N P_\nu P_{\nu+1} (P_n = P_0)$. On each of these we erect a triangular pyramid $Q_\nu N P_\nu P_{\nu+1} (OQ_\nu$ is perpendicular to the base $N P_\nu P_{\nu+1}$ and $Q_\nu \in S)$ and denote by Π'^*_n the polyhedral surface formed of the lateral surfaces of all the n triangular pyramids that have just been added. We ask the question:

(4.1) <u>Under what conditions is the surface</u> Π'^* <u>convex</u>?

<u>Remark 1</u>. The surface $\Pi_n'^*$ is much like the polyhedral surface which H. A. Schwarz inscribed in a cylinder (see his note [3]). In fact if we keep the angle $\alpha = \sphericalangle NOP_0$ fixed and let n become large, it is clear that $\Pi_n'^*$ will be non-convex. The surface $\Pi_n'^*$ is then so tightly corrugated that its area will tend to infinity as $n \to \infty$. This will be the case even if we let $\alpha = \alpha_n \to 0$, but not too fast.

<u>Remark 2</u>. Of interest is the special case when $n = 3$ and Π_3 is the regular tetrahedron T inscribed in S. In this case we have

(4.2) $$\alpha = \alpha_0 \approx 109°\,28'\,16'' \text{ determined by } \cos\alpha_0 = -1/3.$$

The easiest way to show that for T we have $\alpha = \alpha_0$ is to use Kepler's remark that the four vertices $A, B, C, D,$ of a cube that are pairwise opposite on its faces, are the vertices of a regular tetrahedron. Such vertices are $A = (1,1,1)$, $B = (-1,-1,1)$, $C = (-1,1,-1)$, $D = (1,-1,-1)$. with $\alpha_0 = \sphericalangle AOB$ we find on using inner products of vectors that

$$\cos\alpha_0 = \vec{OA} \cdot \vec{OB}/(OA \cdot OB) = -1/(\sqrt{3} \cdot \sqrt{3}) = -1/3 .$$

We may now state our

<u>Theorem 3</u>. <u>The lateral surface</u> $\Pi_n'^*$ <u>is convex if and only if</u>

(4.3) $$n = 3 \text{ and } \alpha \leqq \alpha_0 .$$

<u>Proof</u>: Let us consider on the unit sphere S a right spherical triangle having the angles $A, B, C,$ and opposite sides $a, b, c,$ right-angled at C, hence $C = \pi/2$. We then have the relations

(4.4) $$\cos c = \cos a \cos b$$

and

(4.5) $$\sin B = \sin b/\sin c \ .$$

(We refer to any book on spherical trigonometry).

We now focus our attention on the two neighboring pyramids of $\Pi_n'^*$: $Q_0 N P_0 P_1$ and $Q_{n-1} N P_{n-1} P_0$. Since all their vertices are on S, we may describe the situation in terms of the spherical Figure 2, where all arcs are arcs of great circles, except the arc $P_{n-1} P_0 P_1$ which is part of a parallel circle of colatitude = α. We join Q_0 to Q_{n-1} by an arc of great circle and let it intersect NP_0 at D. From the right spherical triangle $D Q_0 N$ and (4.4), (4.5), we obtain the relations

$$\cos r = \cos d \cos \alpha/2, \quad \sin \pi/n = \sin d/\sin r \ .$$

Eliminating r between these relations by $\cos^2 r + \sin^2 r = 1$, we find that

(4.6) $$\cos^2 d = \cos^2 \pi/n/(1 - \sin^2 \alpha/2 \sin^2 \pi/n) \ .$$

Let U and V be the midpoints of the segments $Q_{n-1}Q_1$ and $N P_0$, respectively. By the symmetry of the entire figure with respect to the plane $N O P_0$, it is clear that the four points O, V, U, D, are collinear. From the plane right triangle $O U Q_0$, where $\measuredangle O U Q_0 = \pi/2$ and $\measuredangle U O Q_0 = d$, $O Q_0 = 1$, we find that

(4.7) $$O U = \cos d = \cos \pi/n/\sqrt{1 - \sin^2 \alpha/2 \sin^2 \pi/n} \ .$$

Finally, from the plane right triangle O V N, where $\measuredangle O V N = \pi/2$, $\measuredangle V O N = \alpha/2$, $O N = 1$, we see that

(4.8) $$O V = \cos \alpha/2 \ .$$

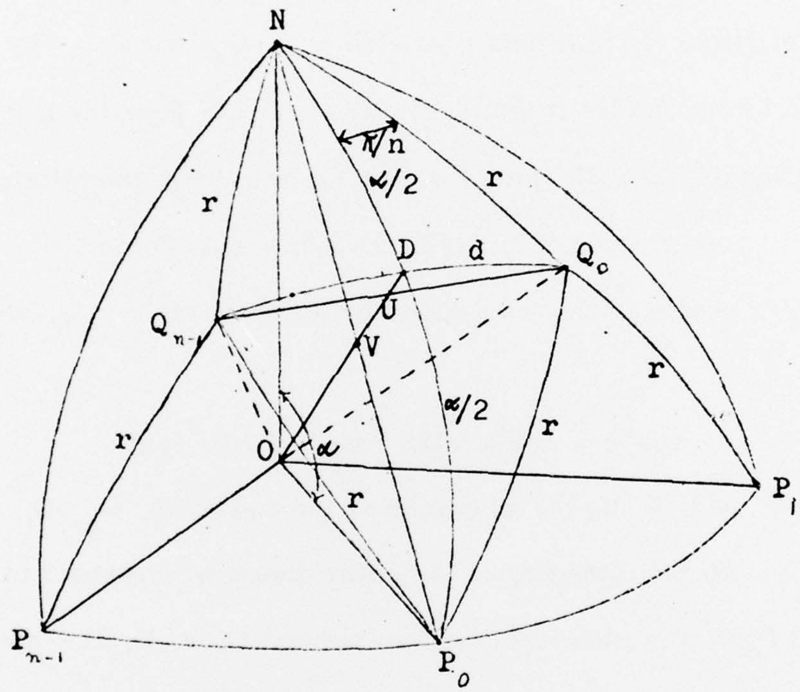

Fig.2

A glance at Figure 2 shows that <u>the surface</u> $\Pi_n'^*$ <u>is non-convex</u> (or corrugated) <u>if and only if the point</u> V <u>is interior to the segment</u> OU, or

$$(4.9) \qquad OV < OU .$$

From (4.7) and (4.8) we find that

$$(4.10) \qquad (OV)^2 = \cos^2 \alpha/2, \; (OU)^2 = \cos^2 \pi/n \big/ (1 - \sin^2 \pi/n \cos^2 \alpha/2) .$$

Writing

$$(4.11) \qquad \xi = \cos^2 \alpha/2$$

and using (4.10) we find the inequality (4.9) to be equivalent to

$$\xi(1 - \xi \sin^2 \pi/n) < 1 - \sin^2 \pi/n ,$$

and this can be written as

$$(4.12) \qquad (1 - 2\xi \sin^2 \pi/n)^2 > (1 - 2 \sin^2 \pi/n)^2 .$$

Observe that if $n = 4$, then $2 \sin^2 \pi/4 = 1$, and that $2 \sin^2 \pi/n < 1$ if $n > 4$. Since $0 < \xi < 1$ by (4.11), it is clear that (4.12) holds if $n \geqq 4$. Therefore

(4.13) <u>the inequality</u> (4.9) <u>holds if</u> $n \geqq 4$.

If $n = 3$ then $2 \sin^2 \pi/3 = 3/2$ and (4.12) is easily found to be equivalent to

$$(\xi - \frac{1}{3})(\xi - 1) > 0 .$$

Since $\xi < 1$, this reduces to $\cos^2 \alpha/2 < 1/3$, or $1 + \cos \alpha < 2/3$, and finally to

$$\cos \alpha < - \frac{1}{3} .$$

In view of $\cos \alpha_0 = -1/3$, our final result for $n = 3$ is this: $\Pi_3^{'*}$ is non-convex if and only if $\alpha > \alpha_0$. It follows that $\Pi_3^{'*}$ is convex if and only if $\alpha \leqq \alpha_0$. This and (4.13) establish Theorem 3.

Remark 3. From Theorem 3 we immediately derive new proofs of Corollaries 1 and 2. Indeed, the four faces of the octahedron O that meet in a vertex N form a lateral surface Π_4'. Likewise, the five faces of the icosahedron I meeting in a vertex form a Π_5'. Since 4 and 5 exceed 3, we conclude by Theorem 3 that $\Pi_4^{'*}$ and $\Pi_5^{'*}$ are corrugated.

In our previous discussion we have left out the base $P_0 P_1 \ldots P_{n-1}$ of the pyramid Π_n and studied only its lateral surface Π_n'. However, we can consider the entire Π_n and ask when Π_n^* is convex, but now we must expressly require that the center O of the sphere S be inside or on the boundary of Π_n. We easily obtain the

Corollary 3. Let Π_n be a regular pyramid inscribed in the sphere S so that its center O is inside Π_n or on its boundary. The polyhedron Π_n^* is convex if and only if

$$n = 3 \quad \text{and} \quad \pi/2 \leqq \alpha \leqq \alpha_0 .$$

The lower bound $\pi/2$ for α is due to the requirement that O belong to Π_n.

The question of the convexity of Π^* can also be settled for another class of simple polyhedra:

Theorem 4. *Let Π be a regular prism, having as base a regular n-gon of radius r, inscribed in the unit sphere S. Then Π^* is convex if and only if*

$$\cos \pi/n \leqq r \leqq 2 \cos \pi/n \Big/ (1 + \cos^2 \pi/n) .$$

We omit the simple proof. As an example let $\Pi = C$ be the cube inscribed in S. Here $n = 4$ and $r = \sqrt{2/3}$ and the inequalities of Theorem 4 are easily verified. Therefore Theorem 4 shows that C^* is convex, a fact that was used in our proof of Corollary 1.

5. A numerical example to § 1

We return to the relations (1.3) of § 1, where P_n is the regular n-gon inscribed in the unit circle C. From (1.3) we readily conclude that

$$\lim_{n \to \infty} \frac{1 - \lambda(P_n)}{1 - \alpha(P_n)} = \frac{1}{4}, \tag{5.1}$$

showing that the approximation of Length C by Length P_n is about four times better than the approximation of Area C by Area P_n.

Does this phenomenon persist if we replace C by a closed convex curve such as an ellipse? We offer here only the numerical example of the ellipse

$$E : x^2 + 2 y^2 = 1 . \tag{5.2}$$

We inscribe in E the closed polygon $P_{24} = A_0 A_1 \cdots A_{24}$, where $A_0 = A_{24} = (1,0)$ and $\sphericalangle A_\nu 0 A_{\nu+1} = 2\pi/24 = 15^\circ$ ($\nu = 0, 1, \ldots, 23$). We find that

(5.3) Length $P_{24} = 5.381384$, Area $P_{24} = 2.191588$,

(5.4) Length $E = 5.402576$, Area $E = 2.221441$,

whence by (1.1) and (1.2)

$$\lambda(P_{24}) = .996077, \qquad \alpha(P_{24}) = .986561 .$$

For the ratio of the relative errors we thus find the value

$$\frac{1 - \lambda(P_{24})}{1 - \alpha(P_{24})} = \frac{.003923}{.013439} = .291912$$

which is not too different from the theoretical limit (5.1) for the circle.

A last word on the derivation of the values (5.4). The second is easy because Area $E = \pi ab$ $(a = 1,\ b = 2^{-\frac{1}{2}})$. The first relation (5.4) requires the numerical evaluation of the elliptic integral

$$\text{Length } E = 4a \int_0^a \sqrt{\frac{a^2 - \epsilon^2 x^2}{a^2 - x^2}}\, dx$$

where $\epsilon = (a^2 - b^2)^{\frac{1}{2}}/a = 1/\sqrt{2}$ is the excentricity of E. Changing variables by setting $x = a \sin \varphi$, the integral becomes

$$\text{Length } E = 4a \int_0^{\pi/2} \sqrt{1 - \epsilon^2 \sin^2 \varphi}\, d\varphi .$$

The numerical value of the last integral with $\epsilon^2 = m = 1/2$, is given to 9 decimal places in [4, Table 17.1, page 609]. The computations leading to the values (5.3) were done with 6 decimal places so that the last two places given may be uncertain.

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In 1898 J. Kürschak observed that the area of a regular 12-gon is $= 3r^2$. This is variously generalized in the plane and in space.

3r square.

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